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STATUS REPORT ON CONTROL OPTIMIZATION, STABILIZATION AND COMPUTER ALGORITHMS FOR SPACE APPLICATIONS

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First Status Report

TO: Office of Research Grants and Contracts (Code SC)
National Aeronautics and Space Administration
Washington, D. C. 20546

SUBJECT: Research Grant No. NGR-22-009 (124)
Control Optimization, Stabilization and Computer Algorithms for Space Applications
First Status Report for Period 1 March through 31 August 1966

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1. GENERAL

This status report covers work carried out by faculty members, research assistants and graduate students associated with the Electronic Systems Laboratory during the period 1 March through 31 August 1966 under Research Grant No. NSG-22-009 (124). Progress was made in two areas: (a) optimal and sub-optimal control; and (b) nonlinear and time-varying systems. This progress is summarized in the next two sections.

During the report period the following people contributed to the project: Professor G. C. Newton, Jr. (part-time), Professor M. Athans (part-time), Professor R. W. Brockett (part-time); A. Debs (from June 15), D. L. Gray (to June 15), M. Gruber, D. L. Kleinman, J. Willems, all full time research assistants; and G. Coune, D. Fitts, U. Forster, I. Gorille, S. Greenberg, R. Gressang, J. Larson, H. S. Witsenhausen, all graduate students (requiring no salary support from project).

This status report does not specifically discuss research plans for the next period since the Technical Proposal of Reference 4 gives this information.

2. SUMMARY OF PROGRESS ON OPTIMAL AND SUB-OPTIMAL CONTROL

The note outlines the research conducted from 1 March 1966 to 1 September 1966 in the area of optimal control and computational algorithms. It should be noted that most of the research was a continuation of our studies initiated and partially supported under NASA Grant NSG-496 with the MIT Center for Space Research.

First we shall describe our research on the development of computational algorithms. Mr. Gray in his doctoral dissertation has considered the problem of the development of computational algorithms for the solution of minimum fuel problems. The motivation for the study arose from the need for computing fuel-optimal controls for attitude corrections of **satelites.** In such problems, one linearizes the equations of motion of the **satelite** to obtain a system of linear and time-invariant differential equations; there are four first-order differential equations which describe the system. Mr. Gray used the Newton-Raphson method to develop a computer algorithm which computes the minimum fuel control, as a function of time, for any given initial conditions. The method was implemented by means of a digital computer program and it was tested for a variety of fourth and sixth order systems, including systems which arise in the **satelite** attitude control problem. The extensive computer results illustrated the feasibility of this method. As a by-product valuable information was gained as to the nature and response of minimum fuel systems as well as in regard to some of the numerical problems which arise. These results will be documented in Mr. Gray's Ph.D. dissertation (which is in the final writing stage) and in a paper for publication which is currently under preparation.

Mr. Kleinman's research deals with the problem of computer storage requirements arising in the implementation of the linear and time-varying feedback system associated with the optimal control of a linear time-varying system with quadratic performance criteria. Such problems arise in the guidance of aerospace vehicles. One linearizes the nonlinear equations of motion about a nominal trajectory and there one uses an optimal linear and time-varying feedback system in order to keep the vehicle near its nominal trajectory. Such problems involve the numerical solution of the matrix Riccati differential equation, backward in time, in order to evaluate the time-varying gains required in the feedback system. If one had a digital computer with a large memory, then one would store the values of the optimal gains at several instants of time. Mr. Kleinman's research efforts deal with the practical aspects of having a digital computer with a limited memory storage capacity. In such a case, one can only store only a small sample of the time-varying gains over the entire control interval and one must reconstruct the time-varying gains by some analog or digital filter. Mr. Kleinman is currently investigating several means for the reconstruction of the time-varying gains as well as the trade-offs involved between the memory capacity of the computer and the performance of the feedback system. This research necessitated the development of several properties of the nonlinear matrix Riccati differential equation which are described in a recent report.[1] At this time the research is focused toward schemes for the efficient reconstruction of the time-varying gains from a limited amount of data. It is estimated that this research will be completed by February 1967.

Of interest to NASA is the research of two fellowship students. Mr. Witsenhausen in his doctoral dissertation[2] considered fundamental problems of control under uncertainty and the use of minimaxtype criteria of performance for their control. Mr. Greenberg in his M. S. thesis[3] considered problems associated with the orbit transfer and rendezvous in circular orbits. The criteria for optimality was the minimization of a linear combination of the consumed fuel and of the elapsed time. He used the "method of averaging" to obtain a sub-optimal guidance law.

Our own research into the problem of computation of optimal controls and the development of iterative computational algorithms for their numerical evaluation have convinced us that for a certain class of problems one should concentrate on the development of computer algorithms for the numerical evaluation of the gains required as well as the input-output characteristics of any required nonlinear function generators. The philosophy and the design ideas behind this approach have been described fully elsewhere[4]. In short, it appears that the numerical evaluation of optimal open-loop controls is of great interest in guidance problems, because one is interested in optimal nominal trajectories. However, problems of attitude control require the design of optimal and sub-optimal feedback control systems, which should be, preferably, implementable using analog hardware. Motivated

1. Superscripts refer to items in References.

by such considerations, Professor Athans and Mr. Debs, supported in part by Mr. Greenberg, have been concentrating since June 1 on the examination of theoretical, practical, and computational questions regarding the nature of optimal and sub-optimal feedback control systems. Analytical feedback control laws have been obtained for some simple nonlinear systems. At present, we are concentrating on the development of ideas and concepts for the design of feedback control systems for high order nonlinear systems, with special emphasis on systems arising in the attitude control of space vehicles.

3. NONLINEAR AND TIME-VARYING SYSTEMS

Progress during the first six months can be reported in four areas:

i) The use of Liapunov theory in the design of magnetic attitude control systems

ii) Stability and Oscillations in nonlinear and time-varying feedback loops

iii) Stability in nonlinear discrete-time feedback systems and applications to the convergence of iterative procedures

iv) *a priori* bounds on the solution of optimization problems

Our results are surveyed below. For more details, see the theses and papers under References.

i) Magnetic Attitude Control

The fact that some types of attitude control of satellites can be achieved using electromagnets which interact with the earth's magnetic field is now widely known. Most treatments of this problem concern spinning satellites and base their analysis on some type of averaging approximation. Although this is quite satisfactory for preliminary studies it is an approximate method and the situations in which it gives satisfactory results are not well understood. For this reason we were motivated to try to use Liapunov methods to establish magnetic attitude control laws which can be shown to be asymtotically stable.

In a recent thesis Coupé formulates a general problem of this type in which three electromagnets and three damping rotors are considered. Since the orientation of the body with respect to the magnetic field is critical in determining the torque levels available, the equations of motion must be expressed in inertial (not body fixed) coordinates. The equations of motion for the body itself are expressible as six first-order equations. In addition, one velocity equation must be added for each rotor present. The resulting equations are much too complicated for direct analysis but by picking a Liapunov function which consists of the "wobble energy" of the satellite and a geometric term it is possible to find a control law (the currents through the electromagnets) that makes the Liapunov functions decrease and the system achieve the desired attitude. Several configurations of rotors and magnets are considered in reference (8) but more work is required. As far as we are aware this is the first rigorous analysis of magnetic attitude control systems of this type although Wheeler at Stanford has recently used Liapunov methods for the case of a single coil in a time-varying field.

ii) Stability and Oscillations

Much work has been done on the problem of stability of nonlinear feedback systems containing a single nonlinear element. The question that is generally asked is: "What are sufficient conditions for the feedback system of Figure 1 to be asymptotically stable in the large? A superficial analysis would consider a linearized "equivalent" system and check the stability properties of this equivalent system. The most common types of linearizations are: the "total" linearization, where the nonlinearity $f(\sigma)$ is replaced by the total gain $K_\sigma = \frac{f(\sigma)}{\sigma}$; and the "local" linearization, where the nonlinearity $f(\sigma)$ is replaced by the incremental gain $K_\sigma = \frac{\partial f(\sigma)}{\sigma}$. It has been hypothesized that the nonlinear system is asymptotically stable in the large if the linearized system is stable forall values of σ. However, we are now able to give theoretical evidence which proves that this approach can lead to incorrect conclusions. The way that these conjectures were disproven was by finding oscillations where linearization predicts none will exist. As is well known, proving the existence of a limit cycle in a high order systems is a hard task. The most important techniques are topological methods, which were used here for this first time in this context, and averaging theory.

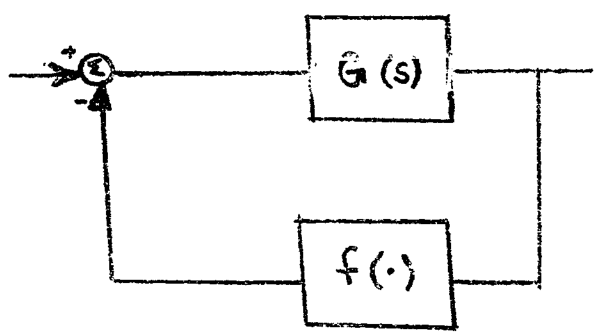

Figure 1. Feedback Systems under consideration

The use of averaging theory to predict when linearization techniques will give the wrong answer about the stability properties of a nonlinear feedback system seems to be very promising. In averaging theory, the nonlinearity is a function of a parameter, ε and for $\varepsilon = 0$ this nonlinearity is zero. Conclusions about the existence of limitcycles can then be made rigorously only in the limit, i.e. when $|\varepsilon| < \varepsilon_0$ for some ε_0. (See Reference 14)

The main theorem (stated here for easy reference) is this:

<u>Theorem</u>: Consider the differential equation:

$$\underline{\dot{z}} = \varepsilon \underline{f}(\underline{z}, t, \varepsilon)$$

$$\underline{f}(\underline{z}, t+T, \varepsilon) = \underline{f}(\underline{z}, t, \varepsilon)$$

and f is a sufficiently smooth function of $\underline{z}$, t and , and define

$$\underline{p}(\underline{a}) = \frac{1}{T}\int_0^T f(\underline{a}, t, 0)dt$$

if there exists an $a_0 \neq \underline{0}$ such that

$$p(\underline{a}_0) = \underline{0}$$

and

$$\det \frac{\partial \underline{p}(\underline{a}_0)}{\partial \underline{a}} \neq 0$$

Then there exists an $\varepsilon_0 > 0$, such that for $0 \leq \varepsilon < \varepsilon_0$, the differential equation has a solution, $\underline{z}(t, \varepsilon)$ with $\underline{z}(t, +T, \varepsilon) = \underline{z}(t, \varepsilon)$ and

$$\lim_{\varepsilon \to 0} \underline{z}(t, \varepsilon) = \underline{a}_0.$$

This technique was applied to the following differential equation:

$$\ddddot{x} + 10\ddot{x} + 9x + (\alpha\dddot{x} + \beta\ddot{x} + \gamma\dot{x} + \delta x) + \varepsilon f(\dot{x}) = 0$$

This is a special case of the feedback system of Figure 1 with

$$G(s) = \frac{s^2}{(s^2+1)(s^2+9) + \varepsilon(\alpha s^3 + \beta s^2 + \gamma s + \delta)}$$

When $\gamma - \alpha > 0$ and $9\alpha - \gamma > 0$, the root locus of this system looks as in Figure 2, and the Nyquist locus of $G(s)$ as in Figure 3. The system is thus stable for all linear negative feedback. It was proven using averaging theory that for $f(\sigma)$ almost any function of σ, this system sustains a limitcycle for ε sufficiently small. Very "reasonable" feedbacks such as $\varepsilon(.)^3$, for which the total gain as well as the incremental gains is always positive and hence for which it seems very reasonable to expect this system to be asymptotically stable in the large, are proven to cause instability.

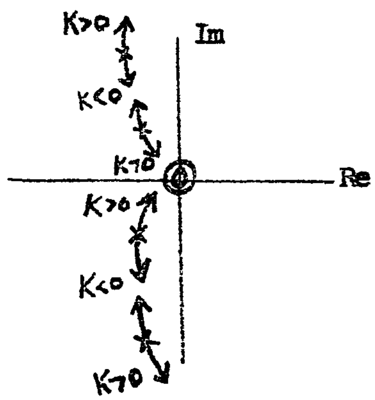

Figure 2: Root locus of example

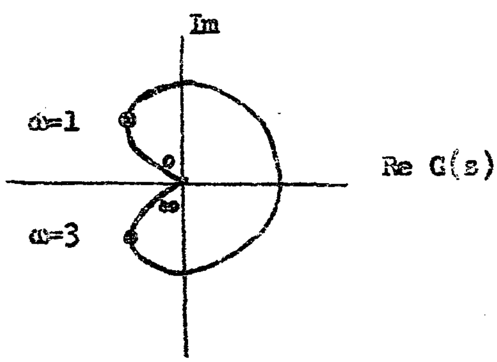

Figure 3: Nyquist locus of example

This particular example as many others, shows that results concerning the stability of nonlinear systems cannot, as one might hope, be derived from linearization and that one has to be extremely careful in drawing conclusions on this basis. Similar results are available for whole classes of systems.

iii) Discrete Time Stability and Convergence of Numerical Procedures

In Gorille's thesis[5], the discrete time stability criteria of Tsypkin, Szego, et al. were used as a tool for establishing the convergence of iterative schemes for finding the location of the real roots of equations of the form

$$f(x) = 0 \quad (x = \text{scalar})$$

The basic iterative procedure was to let $x(k+1)$ be given by

$$x(k+1) = f[x(k)]$$

and this in turn was modified by adding acceleration terms much as one would insert lead compensations in a first order servo system. Detailed numerical results showing the effects of acceleration terms are now available but as yet we cannot claim to have anything like a general theory. The most that one can say at present is that in some cases the improvement obtained using "compensation" is substantial.

The principal source difficulty in this area seems to be a lack of precisely stated results on discrete-time stability and the absence in the literature of any thorough comparison between the criteria of Tsypkin and Szego et al. Some results in this direction are contained in Gorille's thesis but more work is required.

iv) A Priori Bounds on Optimization Problems

The question of just how much feedback control can improve the performance of a system is answered for a particular class of systems in the work reported below. The particular result reported here was discovered somewhat by chance but ties in closely with our work on frequency domain methods.

Consider a controllable and observable system

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{b}u; \quad y = \underline{c}'\underline{x}$$

where u and y are scalars. Suppose that the eigenvalues of $\underline{A}$ lie in the half-plane Re $s < 0$. Under this assumption we can show that the optium value of the performance measure

$$q = \int_0^\infty u^2(t) + y^2(t)dt$$

corresponding to any given initial state $\underline{x}_0$ is related to the "uncontrolled" value of q, i.e.

$$q_0 = \int_0^\infty \underline{x}_0' e^{\underline{A}'t} \underline{cc}' e^{\underline{A}t} \underline{x}_0 dt$$

by the inequality

$$q/q_0 \leq 1/(1+\mu^2)$$

where

$$\mu = \max_\omega \left| g(j\omega) \right| = \max_\omega \left| \underline{c}'(Ij\omega - \underline{A})^{-1} \underline{b} \right|$$

The importance of this result is that it gives some indication of how much benefit one can expect from feedback <u>simply by looking at the peak amplitude of the Bode Plot</u>.

<u>Sketch of Proof</u>: Since the problem is linear we can, without loss of generality, assume that $q_0 = 1$. Then for a given u and $\underline{x}_0$

$$q/q_0 = \int_0^\infty u^2 + \left[\underline{c}' e^{\underline{A}t}(\underline{x}_0 + \int_0^t e^{-\underline{A}\sigma} u(\sigma)d\sigma\right]^2 dt$$

$$= \int_0^\infty u^2 dt + (\underline{c}' e^{\underline{A}t} \underline{x}_0)^2 + 2\underline{c}' e^{\underline{A}t} x_0 \int_0^\infty \underline{c}' e^{A(t-\sigma)} bu(\sigma)d\sigma$$

$$+ \left[\int_0^\infty c' e^{\underline{A}(t-\sigma} \underline{} bu(\sigma)d\sigma\right]^2 dt$$

Now by assumption integral of the second term has a value of 1. Thus

$$q/q_0 = 1 + \int_0^\infty u^2 + 2\underline{c}' e^{\underline{A}t} \underline{x}_0 g(t) + g^2(t)dt$$

where

$$g(t) = \int_0^t \underline{c}' e^{\underline{A}(t-\sigma)} \underline{b} u(\sigma)d\sigma$$

From a well known result on the L_2 norm of linear constant systems, it follows that

$$\rho = \left[\int_0^\infty g^2(t)dt\right]^{1/2} \leq \max_{\omega} \; c'(\underline{I}j\omega-\underline{A})^{-1}\underline{b}\; \eta$$

where

$$\eta = ||u||_2 = \left[\int_0^\infty u^2(t)dt\right]^{1/2}$$

The Cauchy-Schwartz inequality gives

$$\left|2\int_0^\infty \underline{c}'e^{\underline{A}t}\underline{x}_0 g(t)dt\right| \leq 2\rho$$

Hence

$$q/q_0 \geq + \eta^2 - 2\rho + \rho^2$$

where

$$\rho = \left[\int_0^\infty g^2(t)dt\right]^{1/2} \leq \mu\eta$$

Minimizing q/q_0 subject to the constraints $\rho \leq \mu\eta$ gives our inequality.

4. PROJECT DOCUMENTATION STATUS

All of the reports and papers listed under "References" are the result of research supported wholly or in part by this Research Grant. As the reprints become available they will be forwarded as part of the project documentation.

5. BUDGET STATUS

As of 31 August 1966, approximately 43 per cent of the total Grant funds have been expended in a time corresponding to 67 per cent of the Grant period. This reflects a lower-than-anticipated note of expenditure during the summer months. It is expected that the expenditure rate will be nearer the budgeted rate during the remaining months.

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